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BOUNDING THE RELIABILITY OF MULTISTATE SYSTEMS.(U)
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BOUNDING THE RELIABILITY OF MULTISTATE SYSTEMS

BY

DAVID A. BUZEN

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AND
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STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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BOUNDING THE RELIABILITY OF MULTISTATE SYSTEMS

David A. Butler

1. Introduction

Computing the exact reliability of a complex system can involve extremely large amounts of computation. Therefore, a number of more easily computed approximations to the system reliability, based upon minimal paths and minimal cuts, have been proposed, [1], [7], [11], [14]. Some of these approximations are conservative in that they underestimate the system reliability, and the others are optimistic in that they overestimate. Thus, each approximation is actually a reliability bound. To further reduce the computations of system reliability to feasible levels, modular decompositions of complex systems have been exploited, both in the calculation of the exact system reliability and its bounds [4].

All of the work mentioned above has been limited to binary coherent systems. Recently a number of investigators have begun to deal with multistate systems, in which components can exist in more than two states, [2], [5], [6], [10], [12], [13]. This report will develop reliability bounds for multistate systems analogous to those mentioned above for binary systems. The development of these results will be similar to that for their binary counterparts, and readers interested in the details of the proofs in Sections 2 and 3 may find it helpful to refer back to the corresponding binary proofs.

The notation used in this report follows, with a few exceptions, that contained in [1], [6]:

i) $\underline{x} = (x_1, \ldots, x_n)$ denotes the vector of component states.

ii) $C = \{1, 2, \ldots, n\}$ denotes the set of component labels.

iii) $S = \{0, 1, \ldots, M\}$ denotes the set of component states, i.e., $x_i \in S$ for all $i \in C$. The states are ordered, with 0 being the worst state and M the best.

iv) The terms "increasing" and "decreasing" are used in the sense of "non-decreasing" and "non-increasing", respectively.

v) $\underline{X} = (X_1, \ldots, X_n)$ denotes a random vector of component states.

vi) $q_{ij} = \Pr\{X_i = j\}$ $\quad i \in C, j \in S.$

vii) $Q_i(j) = \Pr\{X_i \geq j\}$ $\quad i \in C, j \in S.$

viii) A "performance-distribution vector" is a vector $\underline{v} = (v_0, v_1, \ldots, v_M)$ satisfying $0 \leq v_i \leq 1$, $v_0 = 1$, and $v_{i-1} \geq v_i$, $i = 1, \ldots, M$. For example, $Q_i(\cdot)$ is such a vector.

ix) A "performance-distribution matrix" is a matrix all of whose rows are performance-distribution vectors. For example, $Q_\cdot(\cdot)$ is such a matrix.

x) $\underline{y} < \underline{x}$ means $y_i \leq x_i$ for all i, and $y_i < x_i$ for some i.

xi) $\coprod_{i=1}^{n} z_i = 1 - \prod_{i=1}^{n} (1 - z_i)$.

xii) $\phi : S^n \to S$. $\phi(\underline{x})$ is the system state, given the vector of component states is $\underline{x}$.

xiii) When $X_1, \ldots, X_n$ are independent, we will write $H_\phi^m(Q) = \Pr\{\phi(\underline{X}) \geq m\}$, and $H_\phi(Q) = (H_\phi^0(Q), H_\phi^1(Q), \ldots, H_\phi^M(Q))$. Note that $H_\phi(Q)$ is a performance-distribution vector.

xiv) Script letters $\mathscr{C}$, $\mathscr{P}$, $\mathscr{V}$, $\mathscr{W}$, $\mathscr{Z}$ will denote partitions of C into $M+1$ sets, e.g., $\mathscr{P} = (P^0, P^1, \ldots, P^M)$ where $P^i \subset C$, $P_i \cap P_j$ is empty for $i \neq j$, and $\bigcup_{j=0}^{M} P^j = C$.

xv) Given a partition $\mathscr{P}$, $\underline{x}(\mathscr{P}) = (x_1(\mathscr{P}), \ldots, x_n(\mathscr{P}))$ will denote a vector in S^n for which $i \in P^{x_i(\mathscr{P})}$ for all i. (Intuitively, $\mathscr{P}$ defines the state of each component by partitioning the set C of components into $M+1$ subsets; $\underline{x}(\mathscr{P})$ is the vector of component states corresponding to $\mathscr{P}$.) There is a one-to-one correspondence between partitions of C and vectors in S^n.

xvi) Given two partitions $\mathscr{V}$ and $\mathscr{W}$, we will define the partition $\mathscr{Z} = \mathscr{V} \vee \mathscr{W}$ by

$$x_i(\mathscr{Z}) = \max\{x_i(\mathscr{V}), x_i(\mathscr{W})\} \qquad i \in C$$

and the partition $\mathscr{P} = \mathscr{V} \wedge \mathscr{W}$ by

$$x_i(\mathscr{P}) = \min\{x_i(\mathscr{V}), x_i(\mathscr{W})\} \qquad i \in C .$$

xvii) $(k_i, \underline{x}) = (x_1, \ldots, x_{i-1}, k, x_{i+1}, \ldots, x_n)$.

There is no agreement on the most appropriate definition of "multistate coherent system". Almost no two definitions agree, the differences ranging from minor to major. Griffith [10] discusses and compares a number of possible definitions. The definition we will use in this report is one proposed earlier by the author [5], and is very similar to one proposed in [10].

Definition 1.1. The ordered pair (C, ϕ) is a multistate coherent system if and only if

i) $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$, where $\underline{M} = (M, \ldots, M)$,

ii) $\phi(\underline{x})$ is increasing in $\underline{x}$,

iii) for every $i = 1, \ldots, n$, there exists an $\underline{x}$ such that $\phi(M_i, \underline{x}) > \phi(0_i, \underline{x})$. □

If some component i naturally has only states $0, \ldots, k$, where $k < M$, one can nonetheless formally consider the component to also have states $k + 1, \ldots, M$, which are operationally identical to state k, and then simply take $q_{i\ell}$ to be zero for $\ell > k$. Thus, systems whose components have varying numbers of natural states can be accommodated in the above definition.

Definition 1.2. The multistate coherent system (A, χ) is a module of the multistate coherent system (C, ϕ) if and only if $A \subset C$ and $\phi(\underline{x}) = \psi(\chi(\underline{x}^A), \underline{x}^{A^C})$, where ψ is a multistate coherent structure function, and $\underline{x}^A$, $\underline{x}^{A^C}$ denote those subvectors of $\underline{x}$ consisting of coordinates which are in A, A^C, respectively. □

Definition 1.3. A modular decomposition of a multistate coherent system (C, ϕ) is a set of disjoint modules $(A_1, \chi_1), \ldots, (A_r, \chi_r)$ together with an organizing structure ψ such that

i) ψ is a multistate coherent structure function,

ii) $A_1, \ldots, A_r$ partitions C,

iii) $\phi(\underline{x}) = \psi(\chi_1(\underline{x}^{A_1}), \ldots, \chi_r(x^{A_r}))$.

□

In the following sections the criterion by which we will assess the system performance is $\Pr\{\phi(\underline{X}) \geq m\}$ where $m = 1, \ldots, M$. This criterion differs from some others which have been proposed, notably $E[\phi(X)]$, or $E[u(\phi(X))]$, [6], [10]. To compare these criteria, consider a very simple example of two systems having states 0, 1, 2. The first system has a 50% chance of working perfectly (being in state 2) and a 50% chance of failing totally (being in state 0). The second always is partially functioning (state 1). Under the $E[\phi(X)]$ criterion, these two systems perform identically, yet the designer who only wanted to avoid total failure would surely prefer the second. At a system level, the design specifications are likely to be of the form "meet or exceed a certain system state at least a certain percentage of time", and so argues in favor of a minimum-level-of-performance criterion. In the following we will speak of "system performance at minimum level m" to describe a criterion of system performance at state m, $m + 1$, ..., or M.

2. Bounding the System Reliability by Means of Paths and Cuts

Using $E[\phi(\underline{X})]$ as the measure of system performance, El-Neweihi, Proschan, and Sethuraman [6] have developed an upper and lower bound for the system performance level. These bounds are derived by comparing the structure ϕ to a "parallel" system and a "series" system built from the same components. Other common bounds for binary coherent systems, such as the max-min bounds [1] , the path-cut bounds [7], and the Bonferroni bounds [11], have not been generalized to multistate systems where $E[\phi(\underline{X})]$ is the system performance criterion. The difficulty in doing this would be to define the notions of paths and cuts which such bounds require. Performance criteria such as $E[\phi(\underline{X})]$, or the more general $E[u(\phi(\underline{X}))]$ ($u(\cdot)$ a utility function) proposed by Griffith [10], do not seem compatible with the notions of paths and cuts. However, notions of paths and cuts can be developed when the minimum-performance-level criterion is used. (It should be noted that with an appropriate choice of $u(\cdot)$, Griffith's criterion of $E[u(\phi(\underline{X}))]$ reduces to the minimum-performance-level criterion.)

Definition 2.1. A partition $\mathscr{C}$ is a cut for system performance at minimum level m if and only if $\phi(\underline{x}(\mathscr{C})) < m$. A cut $\mathscr{C}$ is a minimal cut for system performance at minimum level m if and only if $\phi(\underline{y}) \geq m$ for all $\underline{y} > \underline{x}(\mathscr{C})$. □

Definition 2.2. A partition $\mathscr{P}$ is a path for system performance at minimum level m if and only if $\phi(\underline{x}(\mathscr{P})) \geq m$. A path $\mathscr{P}$ is a minimal

path for system performance at minimum level m if and only if $\phi(\underline{y}) < m$ for all $\underline{y} < \underline{x}(\mathscr{P})$. □

Readers wishing a more complete discussion of these two definitions are referred to [5].

In much of the following we will for the sake of brevity simply refer to min cuts and min paths when the minimum system-performance level is clear from the context. Also, in light of the one-to-one correspondence between a partition $\mathscr{C}$ and the vector $\underline{x}(\mathscr{C})$, we will interchangeably refer to min cuts by $\mathscr{C}$ or $\underline{x}(\mathscr{C})$, and to min paths by $\mathscr{P}$ or $\underline{x}(\mathscr{P})$.

Consider a multistate coherent system (C, ϕ) with min paths $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_s$ and min cuts $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_t$ for its performance at minimum level m. Define the functions $\rho_j(\underline{x})$ and $\kappa_j(\underline{x})$ by means of the following equalities

$$\rho_j(\underline{x}) = I_{\{\underline{x} \geq \underline{x}(\mathscr{P}_j)\}} = \prod_{i=1}^{n} I_{\{x_i \geq x_i(\mathscr{P}_j)\}} \qquad j = 1, \ldots, s, \qquad (2.1)$$

$$\kappa_j(\underline{x}) = I_{\{\underline{x} \not\geq \underline{x}(\mathscr{C}_j)\}} = 1 - I_{\{\underline{x} \leq \underline{x}(\mathscr{C}_j)\}}$$

$$= 1 - \prod_{i=1}^{n} I_{\{x_i \leq x_i(\mathscr{C}_j)\}} \qquad j = 1, \ldots, t. \qquad (2.2)$$

The function $\rho_j(\underline{x})$ is called the minimal-path structure function corresponding to $\mathscr{P}_j$, and $\kappa_j(\underline{x})$ is called the minimal-cut structure function corresponding to $\mathscr{C}_j$. As the following theorem shows, the functioning of the system at minimum level m is determined by either its minimal paths or its minimal cuts.

Theorem 2.1.

$$\text{i)} \quad I_{\{\phi(\underline{x})\geq m\}} = \coprod_{j=1}^{s} \rho_j(\underline{x}) = \max_{1\leq j\leq s} \{\rho_j(\underline{x})\} \tag{2.3}$$

$$\text{ii)} \quad I_{\{\phi(\underline{x})\geq m\}} = \prod_{j=1}^{t} \kappa_j(\underline{x}) = \min_{1\leq j\leq t} \{\kappa_j(\underline{x})\} \tag{2.4}$$

The proof follows readily from Definitions 2.1 and 2.2. □

To establish bounds on the system reliability based on min paths and min cuts, we will require the following lemmas, due to Esary, Proschan, and Walkup [8].

Lemma 2.1. If $T_1, T_2, \ldots, T_n$ are associated random variables, then

$$\Pr\{T_1 > t_1, \ldots, T_n > t_n\} \geq \prod_{i=1}^{n} \Pr\{T_i > t_i\} \tag{2.5}$$

and

$$\Pr\{T_1 \leq t_1, \ldots, T_n \leq t_n\} \geq \prod_{i=1}^{n} \Pr\{T_i \leq t_i\} \tag{2.6}$$

for any choice of $t_1, \ldots, t_n$. □

Lemma 2.2. If $\underline{T} = (T_1, T_2, \ldots, T_n)$ is associated and $f_1(\underline{T}), \ldots, f_k(\underline{T})$ are increasing functions of $\underline{T}$, then $f_1(\underline{T}), \ldots, f_k(\underline{T})$ are associated. □

Theorem 2.2. (Path-Cut Bounds). Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be associated. Then

$$\prod_{j=1}^{t} \Pr\{\kappa_j(\underline{X}) = 1\} \leq \Pr\{\phi(\underline{X}) \geq m\} \leq \coprod_{j=1}^{s} \Pr\{\rho_j(\underline{X}) = 1\} \qquad (2.7)$$

Proof. $\kappa_1(\underline{X}), \ldots, \kappa_t(\underline{X})$ are increasing functions of $\underline{X}$ and so by Lemma 2.2 are themselves associated. Hence, by Lemma 2.1 and Theorem 2.1, and the fact that $\kappa(\cdot)$ is a binary-valued function,

$$\Pr\{\phi(\underline{X}) \geq m\} = \Pr\{\kappa_1(\underline{X}) > 0, \ldots, \kappa_t(\underline{X}) > 0\}$$

$$\geq \prod_{j=1}^{t} \Pr\{\kappa_j(\underline{X}) = 1\} . \qquad (2.8)$$

To establish the upper bound, use Theorem 2.1 to write

$$\Pr\{\phi(\underline{X}) \geq m\} = 1 - \Pr\{\phi(\underline{X}) < m\}$$

$$= 1 - \Pr\{\rho_1(\underline{X}) \leq 0, \ldots, \rho_s(\underline{X}) \leq 0\} . \qquad (2.9)$$

Since $\rho_1(\underline{X}), \ldots, \rho_s(\underline{X})$ are increasing functions of $\underline{X}$ and therefore by Lemma 2.2 associated, we can apply Lemma 2.1 to obtain

$$\Pr\{\phi(\underline{X}) \geq m\} \leq 1 - \prod_{j=1}^{s} \Pr\{\rho_j(\underline{X}) \leq 0\}$$

$$= \coprod_{j=1}^{s} \Pr\{\rho_j(\underline{X}) = 1\} . \qquad (2.10)$$

□

For convenience, take $Q_i(M+1) = 0$ and the product over an empty set of indices to be one.

Corollary 2.1. (Path-Cut Bounds for Independent Components). Let $X_1, \ldots, X_n$ be independent. Then

$$\prod_{j=1}^{t}\left[1 - \prod_{k=0}^{M} \prod_{i\in C_j^k} (1 - Q_i(k+1))\right] \leq H_\phi^m(Q)$$

$$\leq \coprod_{j=1}^{s} \prod_{k=0}^{M} \prod_{i\in P_j^k} Q_i(k) \qquad (2.11)$$

Proof. By the definition of ρ_j, and the independence of $X_1, \ldots, X_n$,

$$\Pr\{\rho_j(\underline{X}) = 1\} = \Pr\left\{\prod_{i=1}^{n} I_{\{X_i \geq x_i(\mathscr{P}_j)\}} = 1\right\}$$

$$= \prod_{i=1}^{n} \Pr\{X_i \geq x_i(\mathscr{P}_j)\} . \qquad (2.12)$$

Since any $i \in C$ is in exactly one subset, say P_j^k, of the partition $\mathscr{P}_j = (P_j^0, \ldots, P_j^M)$

$$\Pr\{X_i \geq x_i(\mathscr{P}_j)\} = \Pr\{X_i \geq k\} = Q_i(k) \qquad \text{when } i \in P_j^k .$$

Thus

$$\Pr\{\rho_j(X) = 1\} = \prod_{i=1}^{n} \Pr\{X_i \geq x_i(\mathscr{P}_j)\} = \prod_{k=0}^{M} \prod_{i \in P_j^k} Q_i(k) \qquad (2.13)$$

and the upper bound follows from Theorem 2.2. The lower bound follows similarly. □

Theorem 2.2 and Corollary 2.1 provide lower bounds based upon cuts and upper bounds based upon paths. We next establish, in Theorem 2.3 and Corollary 2.2 below, lower bounds based upon paths and upper bounds based upon cuts.

Theorem 2.3. (Max-Min Bounds).

$$\max_{1 \leq j \leq s} \{\Pr\{\rho_j(\underline{X}) = 1\}\} \leq P\{\phi(\underline{X}) \geq m\} \leq \min_{1 \leq j \leq t} \{\Pr\{\kappa_j(\underline{X}) = 1\}\} . \qquad (2.14)$$

Proof. By Theorem 2.1, for any choice of $j = 1, \ldots, s$ and $j' = 1, \ldots, t$

$$\rho_j(\underline{X}) \leq I_{\{\phi(\underline{X}) \geq m\}} \leq \kappa_{j'}(\underline{X}) \qquad (2.15)$$

Hence

$$\Pr\{\rho_j(\underline{X}) = 1\} \leq \Pr\{\phi(\underline{X}) \geq m\} \leq \Pr\{\kappa_{j'}(\underline{X}) = 1\} \tag{2.16}$$

for all $1 \leq j \leq s$ and $1 \leq j' \leq t$. The result follows by maximizing over j and minimizing over j'. □

The above result holds regardless of whether or not the components are independent or even associated. When the components are associated we have the following.

Corollary 2.2. (Max-Min Bounds for Associated Components) If $\underline{X}$ is associated, then

$$\max_{1\leq j\leq s}\left\{\prod_{k=0}^{M}\prod_{i\in P_j^k} Q_i(k)\right\} \leq P\{\phi(\underline{X}) \geq m\}$$

$$\leq \min_{1\leq j\leq t}\left\{1 - \prod_{k=0}^{M}\prod_{i\in C_j^k}[1 - Q_i(k+1)]\right\} \tag{2.17}$$

Proof. By Lemma 2.2, the random variables $I_{\{X_i \geq x_i(\mathscr{F}_j)\}}$, $i = 1, 2, \ldots, n$ are associated. From Equation 2.12 and Lemma 2.1,

$$\Pr\{\rho_j(\underline{X}) = 1\} = \Pr\left\{\prod_{i=1}^{n} I_{\{X_i \geq x_i(\mathcal{P}_j)\}} > 0\right\}$$

$$\geq \prod_{i=1}^{n} \Pr\{X_i \geq x_i(\mathcal{P}_j)\} \quad . \tag{2.18}$$

Now applying the second equality in Equation 2.13

$$\Pr\{\rho_j(\underline{X}) = 1\} \geq \prod_{k=0}^{M} \prod_{i \in P_j^k} Q_i(k) \tag{2.19}$$

Applying this inequality in Theorem 2.3 yields the lower bound. The upper bound follows similarly. □

The Bonferroni inequalities [9] can also be used to establish reliability bounds based upon paths and cuts.

Lemma 2.3. (Bonferroni Inequalities). For arbitrary events $E_1, \ldots, E_\ell$,

i) $$\Pr(E_1 \cup E_2 \cup \cdots \cup E_\ell) \leq \sum_{j=1}^{\ell} \Pr(E_j) \tag{2.20}$$

ii) $$\Pr(E_1 \cup E_2 \cup \cdots \cup E_\ell) \geq \sum_{j=1}^{\ell} \Pr(E_j) - \sum_{\substack{j,k=1 \\ j<k}}^{\ell} \Pr(E_j \cap E_k) \quad . \tag{2.21}$$

Applying the Bonferroni inequalities to the events $\{\rho_j(\underline{X}) = 1\}$, $j = 1, \ldots, s$, and $\{\kappa_j(X) = 0\}$, $j = 1, \ldots, t$ yields two more sets of lower and upper bounds on the system reliability.

Theorem 2.4. (Bonferroni Bounds).

$$\text{i)} \quad \Pr\{\phi(\underline{X}) \geq m\} \leq \sum_{j=1}^{s} \Pr\{\rho_j(\underline{X}) = 1\} \tag{2.22}$$

$$\text{ii)} \quad \Pr\{\phi(\underline{X}) \geq m\} \geq \sum_{j=1}^{s} \Pr\{\rho_j(\underline{X}) = 1\}$$

$$- \sum_{\substack{j,k=1 \\ j<k}}^{s} \Pr\{\rho_j(\underline{X}) = \rho_k(\underline{X}) = 1\} \tag{2.23}$$

$$\text{iii)} \quad \Pr\{\phi(\underline{X}) \geq m\} \geq 1 - \sum_{j=1}^{t} \Pr\{\kappa_j(\underline{X}) = 0\} \tag{2.24}$$

$$\text{iv)} \quad \Pr\{\phi(\underline{X}) \geq m\} \leq 1 - \sum_{j=1}^{t} \Pr\{\kappa_j(\underline{X}) = 0\}$$

$$+ \sum_{\substack{j,k=1 \\ j<k}}^{t} \Pr\{\kappa_j(\underline{X}) = \kappa_k(\underline{X}) = 0\} . \tag{2.25}$$

□

The above theorem holds regardless of whether or not the X_j's are independent or associated. Under independence we have

Corollary 2.3. (Bonferroni Bounds for Independent Components). If $X_1, \ldots, X_n$ are independent, then

$$\text{i)}\quad H_{\phi}^{m}(Q) \le \sum_{j=1}^{s} \prod_{k=0}^{M} \prod_{i \in P_j^k} Q_i(k), \tag{2.26}$$

$$\text{ii)}\quad H_{\phi}^{m}(Q) \ge \sum_{j=1}^{s} \prod_{k=0}^{M} \prod_{i \in P_j^k} Q_i(k) - \sum_{\substack{j,\ell=1 \\ j<\ell}}^{s} \prod_{k=0}^{M} \prod_{i \in V^k} Q_i(k) \tag{2.27}$$

where $\mathcal{V} = (V^0, \ldots, V^M) = \mathcal{P}_j \vee \mathcal{P}_\ell$,

$$\text{iii)}\quad H_{\phi}^{m}(Q) \ge 1 - \sum_{j=1}^{t} \prod_{k=0}^{M} \prod_{i \in C_j^k} (1 - Q_i(k+1)), \tag{2.28}$$

$$\text{iv)}\quad H_{\phi}^{m}(Q) \le 1 - \sum_{j=1}^{t} \prod_{k=0}^{M} \prod_{i \in C_j^k} (1 - Q_i(k+1))$$

$$+ \sum_{\substack{j,\ell=1 \\ j<\ell}}^{t} \prod_{k=0}^{M} \prod_{i \in W^k} (1 - Q_i(k+1)) \tag{2.29}$$

where $\mathcal{W} = (W^0, \ldots, W^M) = \mathcal{C}_j \wedge \mathcal{C}_\ell$.

Proof. Equation 2.13 and Theorem 2.4 establish part (i) of the theorem; part (iii) follows similarly. To establish part (ii), it suffices to show that

$$\Pr\{\rho_j(\underline{X}) = \rho_\ell(\underline{X}) = 1\} = \prod_{k=0}^{M} \prod_{i \in V^k} Q_i(k) . \tag{2.30}$$

Now by the definition of $\mathcal{Y}$ and the independence of $X_1, \ldots, X_n$

$$\Pr\{\rho_j(\underline{X}) = \rho_\ell(\underline{X}) = 1\} = \Pr\left\{\prod_{i=1}^{n} I_{\{X_i \geq x_i(\mathcal{P}_j) \vee x_i(\mathcal{P}_\ell)\}}\right\}$$

$$= \Pr\left\{\prod_{i=1}^{n} I_{\{X_i \geq x_i(\mathcal{Y})\}}\right\} = \prod_{k=0}^{M} \prod_{i \in V^k} Q_i(k) \ . \tag{2.31}$$

Part (iv) follows similarly. □

The bounds established in Theorems 2.2, 2.3, and 2.4, and Corollaries 2.1, 2.2, and 2.3 are direct generalizations of the path-cut bounds for binary systems developed by Esary and Proschan [7], the max-min bounds for binary systems developed by Barlow and Proschan [1], and the Bonferroni bounds developed by Messinger and Shooman [11].

As for binary systems, the multistate max-min lower bound seems better than the path-cut lower bound (i.e., provides a tighter bound) for systems composed of very unreliable components, and the path-cut lower bound seems better for systems composed of very reliable components. Similarly, the max-min upper bound seems better than the path-cut upper bound for systems of highly reliable components, and the path-cut upper bound seems better for systems of very unreliable components. Regardless of which of these two bounds is likely to do better for a particular system, one probably should calculate both; since both bound pairs

involve the calculation of the same quantities, there is very little extra work in also computing the second pair while computing the first. Of course, if one is only computing lower bounds (or only upper bounds) this comment does not apply.

The path-based Bonferroni bounds (parts (i) and (ii) of Theorem 2.4 and Corollary 2.3) are most useful when component reliabilities are low, and the cut-based bounds (parts (iii) and (iv) of Theorem 2.4 and Corollary 2.3) are most useful when component reliabilities are high. In intermediate cases these bounds may provide no useful information, since they may take values outside the range [0, 1].

Example 2.1. Consider the following three-state coherent system (C, ϕ) consisting of 10 independent components, and having min-cut vectors as follows:

$\phi(\cdot) \geq 1$ ($m = 1$)	$\phi(\cdot) \geq 2$ ($m = 2$)
(1, 0, 2, 0, 0, 1, 0, 1, 1, 1)	(1, 0, 2, 0, 2, 2, 2, 2, 0, 2)
(1, 0, 2, 0, 0, 1, 0, 1, 2, 0)	(1, 0, 2, 0, 2, 2, 2, 2, 2, 1)
(1, 0, 2, 0, 2, 0, 0, 0, 1, 1)	(1, 1, 1, 1, 2, 2, 2, 2, 0, 2)
(1, 0, 2, 0, 2, 0, 0, 0, 2, 0)	(1, 1, 1, 1, 2, 2, 2, 2, 2, 1)
(1, 1, 1, 1, 0, 1, 0, 1, 1, 1)	(2, 2, 0, 2, 2, 2, 2, 2, 0, 2)
(1, 1, 1, 1, 0, 1, 0, 1, 2, 0)	(2, 2, 0, 2, 2, 2, 2, 2, 2, 1)
(1, 1, 1, 1, 2, 0, 0, 0, 1, 1)	(2, 2, 2, 2, 0, 1, 0, 1, 0, 2)
(1, 1, 1, 1, 2, 0, 0, 0, 2, 0)	(2, 2, 2, 2, 0, 1, 0, 1, 2, 1)
(2, 2, 0, 2, 0, 1, 0, 1, 1, 1)	(2, 2, 2, 2, 2, 0, 0, 0, 0, 2)
(2, 2, 0, 2, 0, 1, 0, 1, 2, 0)	(2, 2, 2, 2, 2, 0, 0, 0, 2, 1)
(2, 2, 0, 2, 2, 0, 0, 0, 1, 1)	(2, 2, 2, 2, 2, 2, 2, 2, 1, 1)
(2, 2, 0, 2, 2, 0, 0, 0, 2, 0)	(2, 2, 2, 2, 2, 2, 2, 2, 2, 0)

We will take the component-performance-distribution matrix, Q, to be

$$Q = \begin{bmatrix} 1.0 & 0.8 & 0.5 \\ 1.0 & 0.8 & 0.4 \\ 1.0 & 0.9 & 0.8 \\ 1.0 & 0.5 & 0.1 \\ 1.0 & 0.6 & 0.5 \\ 1.0 & 0.9 & 0.8 \\ 1.0 & 0.7 & 0.6 \\ 1.0 & 0.5 & 0.1 \\ 1.0 & 0.7 & 0.4 \\ 1.0 & 0.6 & 0.2 \end{bmatrix}.$$

Calculated reliabilities and reliability bounds are as follows. (Computations were done on a computer since a direct computation of $H_{\phi}(Q)$, for example, would be very difficult to do by hand.)

$m = 1$ $\quad \Pr\{\phi(X) \geq 1\} = .996729$

	Lower	Upper
Path-Cut Bounds	.993449	.999269
Max-Min Bounds	.8	.998963
Bonferroni Path Bounds	-6.0478	4.82
Bonferroni Cut Bounds	.99343	.999662

$m = 2$ $\quad \Pr\{\phi(X) \geq 2\} = .317652$

	Lower	Upper
Path-Cut Bounds	.237988	.900234
Max-Min Bounds	.14	.52
Bonferroni Path Bounds	-9.49076	2.22278
Bonferroni Cut Bounds	-.14466	.539065

Note that some of the Bonferroni bounds are outside of the region [0,1].

3. Bounding the System Reliability Using Modular Decompositions

The concept of a module, a subset of the components of a system having its own structure function and acting like a single component in relation to the rest of the system, is well known in reliability [3]. For large, complex systems it is almost inevitable that modules will exist, simply because it seems to be inherent in man's way of thinking to break a large problem into a number of smaller problems. In system design this translates into sequentially breaking a complex machine into a number of systems, each system into subsystems, each subsystem into a number of components, etc. This results in a hierarchy of modular decompositions of the overall machine. There are two important reasons to consider exploiting the modular decompositions of a system in developing reliability bounds. The first, and probably the most important from a practical point of view, is that the calculation of all the min paths or all the min cuts for a very complex system can be incredibly involved and time consuming. For example, the so-called "Rasmussen report" [15] on the assessment of accident risks in U.S. commercial nuclear power plants includes two appendices of approximately 550 pages each, which are entirely devoted to descriptions and listings of the fault-tree models used for analyzing boiling-water and pressurized-water reactors. To calculate directly all the min cuts or all the min paths for such fault-trees is not feasible.

By exploiting modular decompositions, the computations involved in computing reliability bounds can be greatly reduced; indeed for

many complex systems, this would be the only practical (from a computational point of view) way to compute reliability bounds. The second reason to use modular decompositions in computing reliability bounds for the system is that the bounds produced are always as good as, and usually better than, the ones resulting from a more direct approach.

The fundamental idea in exploiting a modular decomposition of a system to obtain bounds on the system reliability is to first determine lower-bound vectors on the performance distribution of each module, and then to treat these lower-bound vectors as if they were actual performance distributions and the modules as if they were components in determining a lower bound for the reliability of the organizing structure which relates the module performances to the system performance. Bodin [4] has shown that for binary systems following such a procedure will result in tighter path-cut bounds than those obtained directly. We will establish similar results for multi-state systems.

Consider a multistate coherent system consisting of independent components, having structure function ϕ, and component performance-distribution matrix Q. Define $L_\phi(Q) = (L_\phi^0(Q), L_\phi^1(Q), \ldots, L_\phi^M(Q))$ as follows:

$$L_\phi^m(Q) = \max_{m \le \ell \le M} \{D_\phi^\ell(Q)\}, \tag{3.1}$$

where

$$D_{\phi}^{0}(Q) = 1 , \tag{3.2}$$

and

$$D_{\phi}^{\ell}(Q) = \prod_{j=1}^{t_\ell} [1 - \prod_{k=0}^{M} \prod_{i \in C_j^k} (1 - Q_i(k+1))] , \tag{3.3}$$

and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{t_\ell}$ in the above equation are the min cuts relative to system performance at minimum level ℓ. The quantity $D_{\phi}^{\ell}(Q)$ is simply the path-cut lower bound for the system reliability relative to performance at minimum level ℓ. One might naturally conjecture that $L_{\phi}^{m}(Q) = D_{\phi}^{m}(Q)$ for all m (which is equivalent to conjecturing that $D_{\phi}^{m}(Q)$ is a decreasing function), but, perhaps surprisingly, this is not so, as the following example shows.

Example 3.1. $M = 2$, $C = \{1, 2, 3\}$

Min-Cut Vectors for System Performance at Minimum Level m=1	Min-Cut Vectors for System Performance at Minimum Level m=2
(1, 0, 0)	(1, 1, 1)
(0, 1, 0)	
(1, 0, 0)	

i	0	1	2
$Q_i(j)$	1	.2	.1

$$D_{\phi}^{1}(Q) = [1 - (1 - .2)^2(1 - .1)]^3 = .076$$

$$D_{\phi}^{2}(Q) = 1 - (1 - .1)^3 = .271$$

Since $L_\phi^m(Q) = \max_{m \le \ell \le M} \{D_\phi^\ell(Q)\} \le \max_{m \le \ell \le M} \{\Pr\{\phi(\underline{X}) \ge \ell\}\} = \Pr\{\phi(\underline{X}) \ge m\}$,

$L_\phi^m(Q)$ is itself a lower bound on the system reliability at minimum performance level m, and is indeed a generally better bound than $D_\phi^m(Q)$. Thus, we have proved the following corollary.

Corollary 3.1. $D_\phi(Q) \le L_\phi(Q) \le H_\phi(Q)$. □

In addition to providing a better lower bound than $D_\phi(Q)$, $L_\phi(Q)$ also has the advantage of being itself a performance-distribution vector and can thus be used in computing system reliability bounds sequentially via modular decompositions.

The following lemma, which will be used in proving the main result of this section, shows that the more reliable the components, the better the path-cut lower bound is on the system reliability.

Lemma 3.1. If $Q \le \widetilde{Q}$, then $L_\phi(Q) \le L_\phi(\widetilde{Q})$.

Proof. Since $Q \le \widetilde{Q}$,

$$D_\phi^\ell(Q) = \prod_{j=1}^{t} [1 - \prod_{k=0}^{M} \prod_{i \in C_j^k} (1 - Q_i(k+1))]$$

$$\le \prod_{j=1}^{t} [1 - \prod_{k=0}^{M} \prod_{i \in C_j^k} (1 - \widetilde{Q}_i(k+1))] = D_\phi^\ell(\widetilde{Q}) \qquad (3.4)$$

for $\ell = 1, \ldots, M$. Thus

$$L_\phi^m(Q) = \max_{m \le \ell \le M} \{D_\phi^\ell(Q)\} \le \max_{m \le \ell \le M} \{D_\phi^\ell(\tilde{Q})\} = L_\phi^m(\tilde{Q}). \tag{3.5}$$

□

Lemma 3.2. Suppose that the coherent structure ϕ can be decomposed into r modules having structure functions $\chi_1, \ldots, \chi_r$, and that the system performance at minimum level m is dictated by the module performances according to a single cut $\mathscr{C} = (C^0, C^1, \ldots, C^M)$, i.e.

$$I_{\{\phi(\underline{x}) \ge m\}} = \coprod_{i=1}^{r} I_{\{\chi_i(\underline{x}) \ge x_i(\mathscr{C})\}} . \tag{3.6}$$

Furthermore, assume that all components are independent. Then

$$D_\phi^m(Q) \le \coprod_{i=1}^{r} D_{\chi_i}^{x_i(\mathscr{C})}(Q) \tag{3.7}$$

Proof. Diagramatically, we can express the system in terms of its modules as follows.

$$\phi: \text{---} \left[\begin{array}{c} \text{---} I_{\{\chi_1(\underline{x}) > x_1(\mathscr{C})\}} \text{---} \\ \text{---} I_{\{\chi_2(\underline{x}) > x_2(\mathscr{C})\}} \text{---} \\ \vdots \\ \text{---} I_{\{\chi_r(\underline{x}) > x_r(\mathscr{C})\}} \text{---} \end{array}\right] \text{---}$$

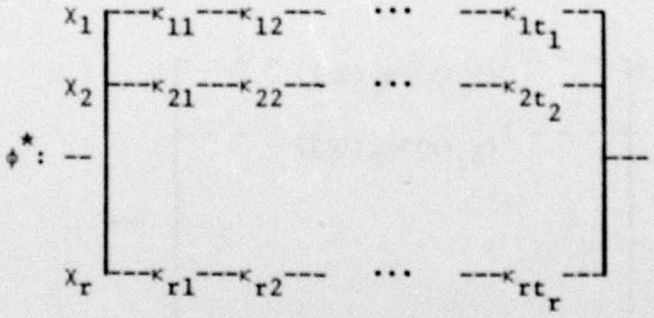
Write $I_{\{X_i(\underline{x}) > x_i(\mathscr{C})\}}$ in terms of its minimal-cut structure functions, $\kappa_{i1}(\underline{x})$, $\kappa_{i2}(\underline{x})$, ..., $\kappa_{it_i}(\underline{x})$, as follows:

$$I_{\{X_i(\underline{x})>x_i(\mathscr{C})\}} = \prod_{j=1}^{t_i} \kappa_{ij}(\underline{x}), \qquad i = 1, \ldots, r . \tag{3.8}$$

Then

$$D_{X_i}^{x_i(\mathscr{C})}(Q) = \prod_{j=1}^{t_i} \Pr\{\kappa_{ij}(\underline{x}) = 1\} \qquad i = 1, \ldots, r \tag{3.9}$$

Now consider a new system ϕ^* obtained from ϕ by replacing each replicated component in each min-cut structure function κ_{ij} by an independent but equally reliable component. Let $\underline{X}^*$ be the random vector of component performances for ϕ^*, and Q^* be the component-performance distribution matrix for ϕ^*. Diagramatically,



By the independence of components in ϕ^*,

$$\Pr\{\phi^*(\underline{X}^*) \geq m\} = \coprod_{i=1}^{r} \prod_{j=1}^{t_i} \Pr\{\kappa_{ij}(\underline{X}) = 1\} \tag{3.10}$$

and so by Equation 3.9

$$H^m_{\phi^*}(Q^*) = \coprod_{i=1}^{r} D^{x_i(\mathscr{C})}_{X_i}(Q) \ . \tag{3.11}$$

Now the min cuts of ϕ and ϕ^* are identical, and each min-cut structure function is obtained by applying the "parallel operator" $\coprod$ to $k_{1j_1}, \ldots, k_{rj_r}$ for some choice of $j_1, \ldots, j_r$ (recall *there is no overlap of components among the various modules*). Thus, $D^m_{\phi^*}(Q^*) = D^m_{\phi}(Q)$. Combining this result with Equation 3.11,

$$\coprod_{i=1}^{r} D^{x_i(\mathscr{C})}_{X_i}(Q) = H^m_{\phi^*}(Q^*) \geq D^m_{\phi^*}(Q^*) = D^m_{\phi}(Q) \ . \tag{3.12}$$

□

Lemma 3.3. If $Q \leq \tilde{Q}$, then $H_{\phi}(Q) \leq H_{\phi}(\tilde{Q})$.

Proof. This lemma is a restatement of Theorem 4.2.i in [6].

We are now in a position to establish the main result of this section.

Theorem 3.1. Let (C, ϕ) be a multistate coherent system of independent components. Suppose (C, ϕ) has a modular decomposition into r modules $\{(A_i, \chi_i)\}$, $i = 1, \ldots, r$, with module organizing structure ψ. Then

$$H_\phi(Q) = H_\psi(\mathcal{H}_{\underline{\chi}}(Q)) \geq \begin{Bmatrix} L_\psi(\mathcal{H}_{\underline{\chi}}(Q)) \\ H_\psi(\mathcal{L}_{\underline{\chi}}(Q)) \end{Bmatrix} \geq L_\psi(\mathcal{L}_{\underline{\chi}}(Q)) \geq L_\phi(Q), \qquad (3.13)$$

where

$$\mathcal{H}_{\underline{\chi}}(Q) = \begin{bmatrix} H_{\chi_1}(Q) \\ \cdot \\ \cdot \\ \cdot \\ H_{\chi_r}(Q) \end{bmatrix} \qquad \mathcal{L}_{\underline{\chi}}(Q) = \begin{bmatrix} L_{\chi_1}(Q) \\ \cdot \\ \cdot \\ \cdot \\ L_{\chi_r}(Q) \end{bmatrix}.$$

Proof: Let $\underline{\chi}(\underline{x}) = (\chi_1(\underline{x}), \ldots, \chi_r(\underline{x}))$. To establish the first equality, note that for any $m = 0, 1, \ldots, M$

$$H_\phi^m(Q) = \Pr\{\phi(\underline{X}) \geq m\} = \sum_{\underline{x} \in S^n} I_{\{\phi(\underline{x}) \geq m\}} \Pr\{\underline{X} = \underline{x}\}$$

$$= \sum_{\underline{x} \in S^n} I_{\{\psi(\underline{\chi}(\underline{x})) \geq m\}} \Pr\{\underline{X} = \underline{x}\}. \qquad (3.14)$$

For any $\underline{y} \in S^r$, let $A(\underline{y}) = \{\underline{x} \in S^n : \underline{\chi}(\underline{x}) = \underline{y}\}$. Since

$$\bigcup_{y \in S^r} A(y) = S^n,$$

$$H_\phi^m(Q) = \sum_{y \in S^r} I_{\{\psi(y) \geq m\}} \sum_{x \in A(y)} \Pr\{\underline{X} = \underline{x}\}$$

$$= \sum_{y \in S^r} I_{\{\psi(y) \geq m\}} \Pr\{\underline{X}(\underline{x}) = \underline{y}\} = H_\psi^m(\mathscr{H}_{\underline{X}}(Q)). \qquad (3.15)$$

To establish the inequalities, note that by applying Corollary 3.1 to the organizing structure ψ and the modules $X_1, \ldots, X_r$, we have

$$H_\psi(\mathscr{H}_{\underline{X}}(Q)) \geq L_\psi(\mathscr{H}_{\underline{X}}(Q))$$

$$H_\psi(\mathscr{D}_{\underline{X}}(Q)) \geq L_\psi(\mathscr{D}_{\underline{X}}(Q))$$

and

$$\mathscr{H}_{\underline{X}}(Q) \geq \mathscr{D}_{\underline{X}}(Q) . \qquad (3.16)$$

Applying Lemmas 3.1 and 3.3 yields

$$H_\psi(\mathscr{H}_{\underline{X}}(Q)) \geq H_\psi(\mathscr{D}_{\underline{X}}(Q)) \quad \text{and} \quad L_\psi(\mathscr{H}_{\underline{X}}(Q)) \geq L_\psi(\mathscr{D}_{\underline{X}}(Q)) . \qquad (3.17)$$

It remains only to show that $L_\psi(\mathscr{D}_{\underline{X}}(Q)) \geq L_\phi(Q)$. Let $\xi_1, \ldots, \xi_t$ denote the minimal-cut structure functions for ψ performing at minimum

level ℓ, and let $\phi_j(\underline{x}) = \xi_j(\chi_1(\underline{x}), \ldots, \chi_r(\underline{x}))$. Let $\mu_{j1}, \ldots, \mu_{jt_j}$ be the minimal-cut structure functions for ϕ_j (performing at minimum level 1, since ϕ_j is binary-valued). Then the collection $\{\mu_{jk} : j = 1, \ldots, t \ \& \ k = 1, \ldots, t_j\}$ are the minimal cuts for ϕ performing at minimum level ℓ, since: $\mu_{jk} = 0$ implies $\phi_j = 0$, which implies $\xi_j = 0$, which implies $\phi = 0$; with j fixed and k varying, the μ_{jk} are distinct, because they are the min cuts for ϕ_j, and for k varying they are distinct because no component is in more than one module; each μ_{jk} is minimal for ϕ since each is minimal for ϕ_j. Thus

$$D_{\phi}^{\ell}(Q) = \prod_{j=1}^{t} \prod_{k=1}^{t_j} \Pr\{\mu_{jk} = 1\} . \tag{3.18}$$

Now by Lemma 3.2

$$D_{\phi_j}^{1}(Q) \leq \coprod_{i=1}^{r} D_{\chi_i}^{x_i(\mathscr{C}_j)}(Q),$$

where $\mathscr{C}_j$ is the min cut corresponding to ξ_j. Since $L_{\chi_i}(Q) \geq D_{\chi_i}(Q)$,

$$D_{\phi_j}^{1}(Q) \leq \coprod_{i=1}^{r} L_{\chi_i}^{x_i(\mathscr{C}_j)}(Q) = H_{\xi_j}^{1}(\mathscr{L}_{\underline{X}}(Q)) . \tag{3.19}$$

Using this result we can write

$$D_{\psi}^{\ell}(\mathscr{L}_X(Q)) = \prod_{j=1}^{t} H_{\xi_j}^{1}(\mathscr{L}_X(Q)) \geq \prod_{j=1}^{t} D_{\phi_j}^{1}(Q)$$

$$= \prod_{j=1}^{t} \prod_{k=1}^{t_j} \Pr\{\mu_{jk} = 1\} \ . \quad (3.20)$$

Combining this result with Equation 3.18 yields

$$D_{\psi}^{\ell}(\mathscr{L}_X(Q)) \geq D_{\phi}^{\ell}(Q) \qquad \ell = 0, 1, \ldots, M \ . \quad (3.21)$$

Finally,

$$L_{\psi}^{m}(\mathscr{L}_X(Q)) = \max_{m \leq \ell \leq M} \{D_{\psi}^{\ell}(\mathscr{L}_X(Q))\} \geq \max_{m \leq \ell \leq M} \{D_{\phi}^{\ell}(Q)\} = L_{\phi}^{m}(Q) \ . \quad (3.22)$$

□

It is also possible to use modular decompositions in establishing upper bounds on the system reliability. Since all results and their proofs are close analogues to those which were developed for the path-cut lower bound, we will simply state without proof the main theorem, omitting the preliminary results.

Let

$$E_{\phi}^{0}(Q) = 1$$

and

$$E_{\phi}^{\ell}(Q) = \coprod_{j=1}^{s_\ell} \prod_{k=0}^{M} \prod_{i \in P_j^k} Q_i(k) \quad (3.23)$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_{s_\ell}$ are the min paths relative to system performance at minimum level ℓ. (The quantity $E_\phi^\ell(Q)$ is simply the path upper bound for the system reliability relative to performance at minimum level ℓ.) Let

$$U_\phi^m(Q) = \min_{0 \le \ell \le m} \{E_\phi^\ell(Q)\} \quad \text{and} \quad U_\phi(Q) = (U_\phi^0(Q), \ldots, U_\phi^M(Q)) . \tag{3.24}$$

It is easily verified that $U_\phi(Q) \ge H_\phi(Q)$, and that $U_\phi(Q)$ is a performance-distribution vector.

Theorem 3.2. Under the same hypotheses as Theorem 3.1,

$$H_\phi(Q) = H_\psi(\mathcal{X}_X(Q)) \le \begin{Bmatrix} U_\psi(\mathcal{X}_X(Q)) \\ H_\psi(\mathcal{U}_X(Q)) \end{Bmatrix} \le U_\psi(\mathcal{U}_X(Q)) \le U_\phi(Q), \tag{3.25}$$

where

$$\mathcal{U}_X(Q) = \begin{bmatrix} U_{X_1}(Q) \\ \vdots \\ U_{X_r}(Q) \end{bmatrix}.$$

□

We will complete this section with an example illustrating the decomposition-based method of calculating system reliability bounds.

Example 3.2. Consider the following three-state coherent system (C, ϕ) consisting of 10 independent components, and having a modular decomposition into 3 modules with structures χ_1, χ_2, χ_3 and organizing structure ψ. The first module consists of components 1, 2, 3, 4, and has min-cut vectors as follows:

$\chi_1(\cdot) \geq 1$ (m=1)	$\chi_2(\cdot) \geq 2$ (m=2)
(0, 1, 0, 1)	(1, 0, 2, 0)
(2, 0, 0, 0)	(1, 1, 1, 1)
	(2, 2, 0, 2)

The second module consists of components 5, 6, 7, 8, and has a structure identical to that of the first module. The third module consists of components 9, 10 and has min-cut vectors:

$\chi_3(\cdot) \geq 1$ (m=1)	$\chi_3(\cdot) \geq 2$ (m=2)
(1, 1)	(0, 2)
(2, 0)	(2, 1)

The organizing structure ψ has min-cut vectors

$\psi(\cdot) \geq 1$ (m=1)	$\psi(\cdot) \geq 2$ (m=2)
(1, 0, 0)	(1, 2, 1)
	(2, 0, 1)
	(2, 2, 0) .

From the min-cuts of χ_1, χ_2, χ_3, and ψ one can compute the min-cut vectors for ϕ, which turn out to be identical to the min-cut vectors of Example 2.1. Therefore, we have found a modular decomposition of that system. Taking the component performance distributions to be identical to those of Example 2.1, we can compute

$$L_{\phi}(Q) = (1.0,\ .993449,\ .237988)$$
$$H_{\phi}(Q) = (1.0,\ .996729,\ .317652)$$
$$U_{\phi}(Q) = (1.0,\ .999269,\ .900234)$$

$$L_{\chi_1}(Q) = (1.0,\ .979308,\ .808830)$$
$$H_{\chi_1}(Q) = (1.0,\ .981200,\ .833000)$$
$$U_{\chi_1}(Q) = (1.0,\ .988336,\ .930811)$$

$$L_{\chi_2}(Q) = (1.0,\ .963724,\ .657930)$$
$$H_{\chi_2}(Q) = (1.0,\ .969400,\ .676000)$$
$$U_{\chi_2}(Q) = (1.0,\ .982612,\ .914354)$$

$$L_{\chi_3}(Q) = (1.0,\ .312000,\ .140000)$$
$$H_{\chi_3}(Q) = (1.0,\ .360000,\ .140000)$$
$$U_{\chi_3}(Q) = (1.0,\ .392000,\ .140000)$$

$$L_{\psi}(\mathscr{Z}_{\chi}(Q)) = (1.0,\ .995229,\ .252572)$$
$$H_{\psi}(\mathscr{Z}_{\chi}(Q)) = (1.0,\ .995229,\ .274072)$$

$$L_{\psi}(\mathcal{H}_X(Q)) = (1.0,\ .996729,\ .300184)$$
$$H_{\psi}(\mathcal{H}_X(Q)) = (1.0,\ .996729,\ .317652)$$
$$U_{\psi}(\mathcal{H}_X(Q)) = (1.0,\ .996729,\ .390005)$$

$$H_{\psi}(\mathcal{U}_X(Q)) = (1.0,\ .999269,\ .370486)$$
$$U_{\psi}(\mathcal{U}_X(Q)) = (1.0,\ .999269,\ .448339)\ .$$

In all cases, $L_{\cdot}(\cdot)$ is identical to $D_{\cdot}(\cdot)$, and $U_{\cdot}(\cdot)$ to $E_{\cdot}(\cdot)$. Note that the two computations of the system reliability, direct and decomposition-based, agree. Also, note that the decomposition-based bounds are indeed tighter than the direct bounds which were computed in Example 2.1 and are repeated here.

A comparison of the computer times required to compute the direct bounds and the decomposition-based bounds for the examples may serve to illustrate the computational advantage of exploiting modular decompositions. All bounds were computed via a single program written in BASIC and running on a Digital Equipment Corporation System 2060 computer. The program used 512.14 CPU-seconds* to compute the path-cut bounds and system reliability directly, and a total of 3.66 CPU-seconds* to compute the path-cut bounds and reliabilities for each module and the organizing structure (i.e., the figures shown in example 3.2).

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*These figures reflect the substantial calculations necessary to determine the min paths from the min cuts for each system.

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This report develops a number of upper and lower bounds for the reliability of multistate systems, that is, systems for which each component may exist in one of a finite number of states. These bounds are based on notions of multistate minimal paths and minimal cuts. The bounds are far more easily computed than the exact system reliability. In order to further reduce the computations required and obtain sharper bounds, a modular-decomposition-based bound is developed for multistate systems.

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